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# Communication situations with a hierarchical player partition

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## Abstract:

In this paper we study situations where agents who are divided in hierarchical classes have restricted possibilities for communication. We introduce a class of allocation rules for these situations inspired by the Myerson value (*Myerson* (1977)) and Shapley values with weight systems (*Kalai* and *Samet* (1988)). It is shown that this new class of allocation rules can be characterized by a consistency property, a fairness criterion, a property based on the hierarchy among the agents, and an efficiency criterion. Furthermore, we show that the consistency property can be dropped from the axiomatic characterization when the fairness criterion is strengthened.

Key words: cooperative games, weighted Shapley values, communication restrictions, hierarchical structure.

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# 1 Introduction

Communication situations are situations in which the agents in a cooperative game have restricted communication possibilities. *Myerson* (1977) studied communication situations where all players are equally powerful. For these situations he introduced *graph-restricted games* and he characterized the *Myerson value*, defined as the *Shapley value* of the graph-restricted game, using efficiency and fairness. Communication situations are studied extensively in the literature since ( see *Nouweland* (1993) and *Borm, Nouweland and Tijs* (1994) for detailed surveys on games with communication restrictions).

Communication situations have a number of interesting applications. However, the solution concepts in the literature cannot be used to study situations in which there is a hierarchy among the agents.

An example of a communication situation with a hierarchical structure appears in the relation between the stockholders of a firm. The asymmetry between the stockholders is evident when one considers large (institutional) stockholders and small (private) stockholders. Furthermore one can consider the airline industry, in which there is a growing number of explicit cooperation agreements. This implies that a certain airline company will sometimes represent itself, while at some other time it will be represented by a form of cooperation. A third example appears in Dutch soccer. The teams in the leagues and the national soccer association have to divide the money that television stations pay for the right to broadcast soccer games. The asymmetry between the teams is a result from the difference in popularity of the teams. While some teams attract a lot of viewers, others will only attract a few. A more theoretical example can be found in economic organizations with a hierarchical structuring of economic agents. One can think of a market where some agents take prices as given, while other agents set prices for certain trade relations. A survey of these models can be found in *Brink and Gilles* (1994).

In this paper we model the asymmetry between the players by means of weight systems consisting of hierarchical classes and weights. *Kalai and Samet* (1988) introduced weighted Shapley values with weight systems, a non-symmetric extension of the Shapley value. We will proceed along the lines set by *Myerson* (1977) and *Kalai and Samet* (1988) and define an extension of the *Myerson value*, the *Myerson value with weight system  $\omega$*  defined as the weighted Shapley value with weight system  $\omega$  of the graph-restricted game.

The organization of the paper is as follows. In section 2 we provide some definitions and we introduce the *Myerson value with weight system  $\omega$* . Then, in section 3 we provide an axiomatic characterization of this rule using a consistency property, a fairness criterion, a property based on the hierarchy among the agents, and an efficiency criterion. In section 4 we introduce a balanced contributions criterion and show that if we replace the

fairness criterion by this balanced contributions criterion, we can drop the consistency property in the axiomatic characterization of section 3. In the appendix we show that in both characterizations the properties are logically independent. We conclude with some remarks with respect to the introduction of a generalization of the *position value* (Borm, Owen and Tijs (1992)) to communication situations with a hierarchical player partition.

## 2 Definitions

Let  $\Omega$  be a universe of players. Throughout this paper we will assume a hierarchy on the players in  $\Omega$  given by a weak ordering (satisfying reflexivity, completeness and transitivity). This ordering will be denoted by  $\Sigma$ .

In this paper we will only consider finite sets of players  $N = \{1, \dots, n\} \subseteq \Omega$ . The hierarchical structure  $\Sigma$  on  $\Omega$  imposes a partition of a player set  $N$  in hierarchical classes,  $\Sigma = (S_1, \dots, S_m)$ , where  $S_1$  denotes the lowest class and  $S_m$  the highest one. Notice that we use  $\Sigma$  to denote both the hierarchical ordering on  $\Omega$  and the partition this structure imposes on  $N$ .

Apart from the hierarchical structure there is a communication structure among the agents. This structure is described by a (communication) graph  $(N, L)$  in which two players are linked iff they can communicate directly. The economic possibilities of a group of players are described by a TU-cooperative game  $(N, v)$ . Our interest in this paper is to investigate the effect that a specific communication structure has on the economic positions of the players. Therefore, we will consider *communication situations*  $(N, v, L)$ , where  $(N, v)$  denotes a TU-cooperative game and  $(N, L)$  denotes a (communication) graph. We will denote the class of all communication situations with player set  $N$  by  $CS^N$ .

For any  $(N, v, L)$  and any  $S \subseteq N$ , the communication possibilities within coalition  $S$  are exactly the links in the set  $L(S) := \{\{i, j\} \in L \mid \{i, j\} \subseteq S\}$ . Hence, a coalition  $S$  is split into communication components as follows:  $T \subseteq S$  is a communication component if and only if the graph  $(T, L(T))$  is connected and there exists no strict superset  $T'$ ,  $T \subset T' \subseteq S$ , with  $(T', L(T'))$  connected. We will denote the resulting partition of  $S$  by  $S/L$ .

In order to capture differences in the relative positions of agents, Kalai and Samet (1988) introduced *weight systems*. A weight system  $\omega$  is a pair  $(\lambda, \Sigma)$  with  $\lambda \in \mathbb{R}_{++}^\Omega$  a set of weights and  $\Sigma$  the hierarchical ordering on  $\Omega$ . Now define for all finite  $N \subseteq \Omega$ ,  $S \subseteq N$  the set  $\overline{S}$  as follows: let  $k := \max\{j \mid S_j \cap S \neq \emptyset\}$  and  $\overline{S} := S \cap S_k$ . So,  $\overline{S}$  denotes the

set of players of  $S$  in the highest hierarchical class that is represented in  $S$ . For every *unanimity game*  $(N, u_S)$ , defined by  $u_S(T) = 1$ , if  $S \subseteq T$ , and  $u_S(T)=0$  otherwise, the *weighted Shapley value with weight system*  $\omega$ ,  $\Phi^\omega$ , assigns to the players of  $\bar{S}$  an amount proportional to the weights of these players and nothing to the other players, so

$$\Phi_i^\omega(u_S) := \begin{cases} \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j} & , \text{ if } i \in \bar{S} \\ 0 & , \text{ otherwise} \end{cases}.$$

Harsanyi (1959) showed that every *characteristic function* can be written in exactly one way as a linear combination of unanimity games:  $v = \sum_{S \subseteq N} \alpha_S^v u_S$ , where  $(\alpha_S^v)_{S \subseteq N}$  are called the dividends of the game  $(N, v)$ . The weighted Shapley value with weight system  $\omega$  of a game  $(N, v)$  is now defined in an additive way by:

$$\Phi_i^\omega(v) := \sum_{S \subseteq N} \alpha_S^v \Phi_i^\omega(u_S).$$

With a communication situation  $(N, v, L)$  we can define the *graph-restricted game*  $(N, v^L)$  (cf. Myerson (1977)). For all  $S \subseteq N$ :

$$v^L(S) := \sum_{C \in S/L} v(C).$$

Myerson (1977) defines the *Myerson value* of a communication situation  $(N, v, L)$  as the Shapley value of the game  $(N, v^L)$ . Analogously we define the *Myerson value with weight system*  $\omega$  of the communication situation  $(N, v, L)$  as the weighted Shapley value with weight system  $\omega$  of the game  $(N, v^L)$ :

$$\mu^\omega(N, v, L) := \Phi^\omega(v^L).$$

Consider the weight system  $\omega^* = ((1, \dots, 1), (N))$ . The weighted Shapley value with weight system  $\omega^*$  of a game coincides with the Shapley value of this game. Therefore, the Myerson value with weight system  $\omega^*$  of a communication situation coincides with the Myerson value of this communication situation. Hence, the Myerson value belongs to the class of Myerson values with weight systems, so, our definition of Myerson values with weight systems is an extension of the definition of Myerson (1977).

### 3 Axiomatic characterization using consistency

In this section we will give an axiomatic characterization of the class of Myerson values with weight systems. This characterization involves a consistency property and three

other axioms, component efficiency, class weighted fairness and higher class independency.

We consider a finite player set  $N \subseteq \Omega$  and the class  $CS^N$  of all communication situations with player set  $N$ . An allocation rule  $\gamma$  is a function on  $\{CS^N \mid N \subseteq \Omega, N \text{ finite}\}$  that assigns to every  $(N, v, L) \in CS^N$ ,  $N$  finite, a vector  $\gamma(N, v, L) \in \mathbb{R}^N$ . When there is no ambiguity about the underlying game  $(N, v)$  we will simply write  $\gamma(L)$  instead of  $\gamma(N, v, L)$ . Examples of allocation rules are the Myerson values with weight systems and the *position value* (see *Borm, Owen and Tijs* (1992)).

We introduce some properties for an allocation rule  $\gamma$ :

**Component efficiency (CE)** : For all finite  $N \subseteq \Omega$ , all communication situations  $(N, v, L) \in CS^N$ , and all  $S \subseteq N$ , if  $S$  is a communication component of  $(N, L)$ , then

$$\sum_{i \in S} \gamma_i(L) = v(S).$$

**Class weighted fairness (CWF)** : There exist weights  $\lambda = (\lambda_i)_{i \in \Omega}$  such that for all finite  $N \subseteq \Omega$ , all communication situations  $(N, v, L) \in CS^N$ , all hierarchical classes  $S_k$ ,  $k \in \{1, \dots, m\}$ , and all  $i, j \in S_k$  it holds that

$$\frac{1}{\lambda_i} (\gamma_i(L) - \gamma_i(L \setminus \{\{i, j\}\})) = \frac{1}{\lambda_j} (\gamma_j(L) - \gamma_j(L \setminus \{\{i, j\}\})).$$

Component efficiency states that the value of a component is divided between the players that form the component. Class weighted fairness states that when we compare a communication situation with the situation that results when we delete a link between two players in the same hierarchical class, then the weighted differences in payoff for these two players are equal.

To prove that Myerson values with weight systems satisfy component efficiency and class weighted fairness we use three of the properties *Kalai* and *Samet* (1988) use to characterize the weighted Shapley values with weight systems, *additivity*, the *dummy player property* and *partnership consistency*.

First we note that  $\Phi^\omega$  satisfies *additivity*, i.e.  $\Phi^\omega(N, v + w) = \Phi^\omega(N, v) + \Phi^\omega(N, w)$  for all  $(N, v)$  and  $(N, w)$ .

A player  $i$  in  $v$  is called a *dummy player* of the game  $(N, v)$  if

$$v(S \cup \{i\}) = v(S) \text{ for all } S \subseteq N \setminus \{i\}.$$

Now,  $\Phi^\omega$  has the *dummy player property*, i.e.  $\Phi_i^\omega(N, v) = 0$  if  $i$  is a dummy player of the game  $(N, v)$ .

A coalition  $S$  is called a *partnership* in  $(N, v)$  if for all  $T \subset S$  and all  $R \subseteq N \setminus S$ ,  $v(R \cup T) = v(R)$ .  $\Phi^\omega$  is *partnership consistent*, i.e. for every *partnership*  $S$  in  $(N, v)$  and the unanimity game  $u_S$ :

$$\Phi_i^\omega(N, v) = \Phi_i^\omega(N, \Phi_S^\omega(N, v)u_S), \text{ for every } i \in S,$$

where  $\Phi_S^\omega(N, v) = \sum_{i \in S} \Phi_i^\omega(N, v)$ . Partnership consistency of  $\Phi^\omega$  states that if we reallocate the total payoff of a partnership according to  $\Phi^\omega$ , then the players receive the same payoff before and after the reallocation.

In lemma 3.1 we will show that  $\mu^\omega$  satisfies the properties component efficiency and class weighted fairness.

**Lemma 3.1** *The Myerson value with weight system  $\omega$  satisfies CE and CWF.*

**Proof:** First we will prove that  $\mu^\omega$  satisfies CE. Let  $N \subseteq \Omega$  be finite,  $(N, v, L) \in CS^N$ , and  $S$  a communication component of  $(N, L)$ . Split the game  $v^L$  into two games  $v^S$  and  $v^{N \setminus S}$ , defined by

$$\begin{aligned} v^S(T) &:= v^L(T \cap S), \\ v^{N \setminus S}(T) &:= v^L(T \setminus S), \end{aligned}$$

for all  $T \subseteq N$ . Since  $S$  is a communication component of  $(N, L)$  it holds that  $v^L = v^S + v^{N \setminus S}$ .

All players of  $S$  are dummy players in the game  $v^{N \setminus S}$ . From the dummy player property of the weighted Shapley value with weight system  $\omega$  it follows that  $\Phi_i^\omega(v^{N \setminus S}) = 0$  for all  $i \in S$  (see *Kalai and Samet (1988)*). In the same way we can show that  $\Phi_i^\omega(v^S) = 0$  for all  $i \in N \setminus S$ . Using the additivity of  $\Phi^\omega$  we obtain

$$\begin{aligned} \sum_{i \in S} \mu_i^\omega(N, v, L) &= \sum_{i \in S} \Phi_i^\omega(v^L) \\ &= \sum_{i \in S} \Phi_i^\omega(v^S) + \sum_{i \in S} \Phi_i^\omega(v^{N \setminus S}) \\ &= \sum_{i \in S} \Phi_i^\omega(v^S) \\ &= \sum_{i \in N} \Phi_i^\omega(v^S) = v^S(N) = v^L(S) = v(S), \end{aligned}$$

where we use the fact that  $\Phi_i^\omega(v^S) = 0$  for all  $i \in N \setminus S$  in the fourth equality and the efficiency of  $\Phi^\omega$  in the fifth equality.

Let  $\omega = (\lambda, \Sigma)$ , then we show that  $\mu^\omega$  satisfies CWF with weights  $\lambda$ . Let  $N \subseteq \Omega$  be finite,  $(N, v, L) \in CS^N$ ,  $k \in \{1, \dots, m\}$ , and  $i, j \in S_k$ . Define  $L' := L \setminus \{\{i, j\}\}$  and

$v^* := v^L - v^{L'}$ . Then for every  $T \subseteq N$  with  $\{i, j\} \not\subseteq T$  it holds that  $T/L = T/L'$ , and, consequently,

$$v^*(T) = \sum_{R \in T/L} v(R) - \sum_{R \in T/L'} v(R) = 0.$$

This implies that  $\{i, j\}$  is a partnership in  $v^*$ . The partnership consistency of the weighted Shapley value with weight system  $\omega$  shown by *Kalai* and *Samet* (1988) gives us:

$$\begin{aligned} \Phi_i^\omega(v^*) &= \Phi_i^\omega\left(\left(\Phi_i^\omega(v^*) + \Phi_j^\omega(v^*)\right)u_{\{i,j\}}\right) \\ &= \frac{\lambda_i}{\lambda_i + \lambda_j} \left(\Phi_i^\omega(v^*) + \Phi_j^\omega(v^*)\right). \end{aligned}$$

Using an analogous expression for  $j$  we find that

$$\frac{\Phi_i^\omega(v^*)}{\lambda_i} = \frac{\Phi_j^\omega(v^*)}{\lambda_j}.$$

This gives us

$$\begin{aligned} \frac{\mu_i^\omega(L) - \mu_i^\omega(L')}{\lambda_i} &= \frac{\Phi_i^\omega(v^L) - \Phi_i^\omega(v^{L'})}{\lambda_i} = \frac{\Phi_i^\omega(v^*)}{\lambda_i} = \\ \frac{\Phi_j^\omega(v^*)}{\lambda_j} &= \frac{\Phi_j^\omega(v^L) - \Phi_j^\omega(v^{L'})}{\lambda_j} = \frac{\mu_j^\omega(L) - \mu_j^\omega(L')}{\lambda_j}, \end{aligned}$$

where the second and the fourth equality follow from the additivity of the weighted Shapley value with weight system  $\omega$ .  $\square$

Before we can introduce the next property, higher class independency, we first need some more definitions. First we define for all  $k \in \{1, \dots, m\}$  the set  $L_{\leq k}$  of links that are between players in classes up to  $S_k$ ,

$$L_{\leq k} := L(\cup_{r=1}^k S_r).$$

Further we define the constants  $(\alpha_S)_{S \subseteq N}$  and  $(\alpha_S^k)_{S \subseteq N}$  such that  $v^L = \sum_{S \subseteq N} \alpha_S u_S$  and  $v^{L_{\leq k}} = \sum_{S \subseteq N} \alpha_S^k u_S$ , i.e.  $(\alpha_S)_{S \subseteq N}$  and  $(\alpha_S^k)_{S \subseteq N}$  are the dividends (see section 2) of  $v^L$  and  $v^{L_{\leq k}}$  respectively. Now we can consider the next property.

**Higher class independency (HCI) :** For all finite  $N \subseteq \Omega$ , every  $(N, v, L) \in CS^N$ , and every  $i \in N$ , if  $i \in S_k$  then  $\gamma_i(N, v, L) = \gamma_i(N, v, L_{\leq k})$ .

Higher class independency states that the payoff of a player is not influenced by breaking all the links in which a player of a higher class is involved. Lemma 3.2 shows that the Myerson value with weight system  $\omega$  satisfies higher class independency.



**Lemma 3.2** *The Myerson value with weight system  $\omega$  satisfies HCI.*

**Proof:** Let  $N \subseteq \Omega$  be finite and  $(N, v, L) \in CS^N$ . Let  $k \in \{1, \dots, m\}$  and  $i \in S_k$  then

$$\mu_i^\omega(N, v, L) = \Phi_i^\omega(v^L) = \sum_{S: i \in \bar{S}} \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j} \alpha_S.$$

Notice that for all  $S \subseteq \cup_{j=1}^k S_j$

$$v^L(S) = \sum_{C \in S/L} v(C) = \sum_{C \in S/L_{\leq k}} v(C) = v^{L_{\leq k}}(S).$$

Using the uniqueness of the dividends  $\alpha_S$  and  $\alpha_S^k$ , we obtain for all  $S \subseteq \cup_{r=1}^k S_r$  that  $\alpha_S = \alpha_S^k$ . It now follows that

$$\mu_i^\omega(N, v, L) = \sum_{S: i \in \bar{S}} \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j} \alpha_S = \sum_{S: i \in \bar{S}} \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j} \alpha_S^k = \mu_i^\omega(N, v, L_{\leq k}),$$

where the second equality uses the fact that if  $i \in S_k$  then for all  $S$  with  $i \in \bar{S}$  it holds that  $S \subseteq \cup_{r=1}^k S_r$ .  $\square$

Before we can introduce the last property that completes our axiomatic characterization of the Myerson values with weight systems, we need some more definitions.

Let  $N \subseteq \Omega$ ,  $N$  finite, and  $(N, v, L) \in CS^N$  be fixed for the moment. First we define the set of links in  $L$ , in which no player in the lowest  $k$  classes is involved:

$$L^{>k} := \{\{i, j\} \in L \mid i, j \in \cup_{r=k+1}^m S_r\}.$$

The players in the highest  $m - k$  classes may be connected via players of the lowest  $k$  classes. These connections are reflected in the set

$$L^{*k} := \{\{i, j\} \mid i, j \in \cup_{r=k+1}^m S_r, \text{ and there exists a } T \subseteq \cup_{r=1}^k S_r, T \neq \emptyset \text{ such that } (T \cup \{i, j\}, L(T \cup \{i, j\})) \text{ is connected}\}.$$

The communication possibilities of the players in the highest  $m - k$  classes are modelled by

$$L^k := L^{>k} \cup L^{*k}.$$

We are mainly interested in  $L^1$  and we will also indicate this set by  $L'$ . So,  $L'$  denotes the adjusted communication possibilities of the players when we leave out the lowest class. Finally we define a game  $(N \setminus S_1, w)$  where for all  $S \subseteq N \setminus S_1$ :  $w(S) := v^L(S \cup S_1) - v^L(S_1)$ .

In lemma 3.3 we prove a relation between the characteristic functions  $w$  and  $w^{L'}$ .

**Lemma 3.3** *Let  $N \subseteq \Omega$ ,  $N$  finite, and  $(N, v, L) \in CS^N$ . Then, it holds that  $w = w^{L'}$ .*

**Proof:** We have to show for all  $S \subseteq N \setminus S_1$  that  $w(S) = w^{L'}(S)$ . Let  $S \subseteq N \setminus S_1$ . From the definition of  $L'$  it follows that for every  $T \in S/L'$  we can define a  $C(T)$  such that  $C(T) \subseteq S_1/L$  and it holds that  $T \cup C'(T) \in (S \cup S_1)/L$ , with  $C'(T) := \cup_{C \in C(T)} C$ . This definition implies that  $C(T) = \emptyset$  if  $T \in (S \cup S_1)/L$ .

We also can conclude from the definition of  $L'$  that for all  $T_1, T_2 \in S/L'$  with  $T_1 \neq T_2$ ,  $C'(T_1) \cap C'(T_2) = \emptyset$ , which states that  $T_1$  and  $T_2$  are in the original situation connected with different components of the lowest class.

Using this we find:

$$\begin{aligned}
w(S) &= v^L(S \cup S_1) - v^L(S_1) \\
&= \sum_{C \in (S \cup S_1)/L} v(C) - \sum_{C \in S_1/L} v(C) \\
&= \sum_{T \in S/L'} v(T \cup C'(T)) + \sum_{C \in S_1/L, \nexists T \in S/L': C \in C(T)} v(C) - \sum_{C \in S_1/L} v(C) \\
&= \sum_{T \in S/L'} v(T \cup C'(T)) - \sum_{C \in S_1/L, \exists T \in S/L': C \in C(T)} v(C),
\end{aligned}$$

where the third equality follows from the fact that every  $C \in C(T)$  is a component in the graph  $(S_1, L(S_1))$ . Since for all  $T \in S/L'$  it holds that  $\sum_{C \in C(T)} v(C) = v^L(C'(T))$  we can rewrite this last expression and obtain

$$\begin{aligned}
&\sum_{T \in S/L'} \{v(T \cup C'(T)) - v^L(C'(T))\} \\
&= \sum_{T \in S/L'} \{v^L(T \cup C'(T)) + v^L(S_1 \setminus C'(T)) \\
&\quad - v^L(C'(T)) - v^L(S_1 \setminus C'(T))\} \\
&= \sum_{T \in S/L'} \{v^L(T \cup S_1) - v^L(S_1)\} \\
&= \sum_{T \in S/L'} w(T) \\
&= w^{L'}(S),
\end{aligned}$$

where the second equality follows because the partition of  $T \cup S_1$  in  $T \cup C'(T)$  and  $S_1 \setminus C'(T)$  does not split up any component of  $(T \cup S_1, L)$  and the partition of  $S_1$  in  $C'(T)$  and  $S_1 \setminus C'(T)$  does not split up any component of  $(S_1, L)$ .  $\square$

The dividends of the game  $(N, w)$  will be denoted by  $(\beta_S)_{S \subseteq N}$ , so  $w = \sum_{S \subseteq N} \beta_S u_S$ . A relation between these dividends and the dividends of the game  $(N, v^L)$ ,  $(\alpha_S)_{S \subseteq N}$ , is expressed in the following lemma.

**Lemma 3.4** *Let  $N \subseteq \Omega$ ,  $N$  finite, and  $(N, v, L) \in CS^N$ . Then, for all  $T \subseteq N \setminus S_1$  with  $T \neq \emptyset$ ,  $\beta_T = \sum_{U \subseteq S_1} \alpha_{T \cup U}$ .*

**Proof:** We will prove the lemma by induction to the number of elements of the set  $T$ . If  $T = \{i\}$  then

$$\begin{aligned} \beta_{\{i\}} &= w(\{i\}) = v^L(\{i\} \cup S_1) - v^L(S_1) = \sum_{S \subseteq \{i\} \cup S_1} \alpha_S - \sum_{S \subseteq S_1} \alpha_S \\ &= \sum_{S = \{i\} \cup U, U \subseteq S_1} \alpha_S = \sum_{U \subseteq S_1} \alpha_{\{i\} \cup U}. \end{aligned}$$

Let  $T \subseteq N \setminus S_1$  with  $|T| > 1$ . Suppose  $\beta_S = \sum_{U \subseteq S_1} \alpha_{S \cup U}$  for all  $S \subseteq N \setminus S_1$  with  $|S| < |T|$ . Notice that

$$w(T) = v^L(T \cup S_1) - v^L(S_1) = \sum_{S \subseteq T \cup S_1} \alpha_S - \sum_{S \subseteq S_1} \alpha_S = \sum_{S \subseteq T \cup S_1, S \cap T \neq \emptyset} \alpha_S.$$

From this and the induction hypothesis it follows that

$$\begin{aligned} \beta_T &= w(T) - \sum_{S \subset T, S \neq \emptyset} \beta_S \\ &= \sum_{S \subseteq T \cup S_1, S \cap T \neq \emptyset} \alpha_S - \sum_{S \subset T, S \neq \emptyset} \sum_{U \subseteq S_1} \alpha_{S \cup U} \\ &= \sum_{S \subseteq T, S \neq \emptyset} \sum_{U \subseteq S_1} \alpha_{S \cup U} - \sum_{S \subset T, S \neq \emptyset} \sum_{U \subseteq S_1} \alpha_{S \cup U} \\ &= \sum_{U \subseteq S_1} \alpha_{T \cup U}, \end{aligned}$$

where the first equality uses the fact that  $\beta_\emptyset = 0$ . □

Using lemma 3.4, we can prove the following statement with respect to the relation between a sum of the dividends of the game  $(N, v)$  and a sum of the dividends of the game  $(N, w)$ :

**Lemma 3.5** *Let  $N \subseteq \Omega$ ,  $N$  finite, and  $(N, v, L) \in CS^N$ . Then, for all  $k \in \{2, \dots, m\}$  and all  $T \subseteq S_k$  with  $T \neq \emptyset$ :*

$$\sum_{S \subseteq \bigcup_{r=1}^{k-1} S_r} \alpha_{T \cup S} = \sum_{S \subseteq \bigcup_{r=2}^{k-1} S_r} \beta_{T \cup S}.$$

**Proof:** Let  $k \in \{2, \dots, m\}$  and  $T \subseteq S_k$  with  $T \neq \emptyset$ . It follows that

$$\sum_{S \subseteq \bigcup_{r=1}^{k-1} S_r} \alpha_{T \cup S} = \sum_{S \subseteq \bigcup_{r=2}^{k-1} S_r} \sum_{U \subseteq S_1} \alpha_{T \cup S \cup U} = \sum_{S \subseteq \bigcup_{r=2}^{k-1} S_r} \beta_{T \cup S},$$

where the second equality follows from lemma 3.4.  $\square$

For the introduction of the property *class consistency* we define the game  $(N \setminus S_1, z)$ , given an allocation rule  $\gamma$ . For every  $S \subseteq N \setminus S_1$  define

$$z(S) := v^L(S \cup S_1) - \sum_{i \in S_1} \gamma_i(S \cup S_1, v, L).$$

The game  $(N \setminus S_1, z)$  is called the *class-reduced game*. The class-reduced game is a specific example of the reduced games *Hart* and *Mas-Colell* (1989) use to characterize the Shapley value. The value of a coalition  $S$  in the class-reduced game is defined as the worth of the union of this coalition with the players of the lowest class in the graph-restricted game minus the payoff that the players of this lowest class would get if the player set was restricted to  $S \cup S_1$ . We will call the triple  $(N \setminus S_1, z, L')$  the *class-reduced communication situation*.

**Class Consistency (CC)** : For all finite  $N \subseteq \Omega$ , all  $(N, v, L) \in CS^N$ , and all  $i \in N \setminus S_1$  it holds that

$$\gamma_i(N, v, L) = \gamma_i(N \setminus S_1, z, L').$$

This property states that to the players that are not in the lowest class an allocation rule  $\gamma$  attributes the same payoff in the class-reduced communication situation as in the original situation.

**Lemma 3.6** *The Myerson value with weight system  $\omega$  satisfies CC.*

**Proof:** Let  $N \subseteq \Omega$  be finite and  $(N, v, L) \in CS^N$ . From CE and HCI of the Myerson values with weight systems (see lemma's 3.1 and 3.2) we immediately find for all  $S \subseteq N \setminus S_1$  that

$$\sum_{i \in S_1} \mu_i^\omega(S \cup S_1, v, L) = \sum_{i \in S_1} \mu_i^\omega(S \cup S_1, v, L_{\leq 1}) = v^L(S_1).$$

So, for every  $S \subseteq N \setminus S_1$  it holds that  $z(S) = w(S)$ . This means that we only have to show that for all  $(N, v, L) \in CS^N$  and all  $i \in N \setminus S_1$

$$\mu_i^\omega(N, v, L) = \mu_i^\omega(N \setminus S_1, w, L').$$

Let  $k \in \{2, \dots, m\}$  and  $i \in S_k$ . Then

$$\mu_i^\omega(N, v, L) = \Phi_i^\omega(v^L) = \sum_{S: i \in \bar{S}} \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j} \alpha_S$$

$$\begin{aligned}
&= \sum_{T \subseteq S_k, i \in T} \left[ \frac{\lambda_i}{\sum_{j \in T} \lambda_j} \sum_{S \subseteq \bigcup_{r=1}^{k-1} S_r} \alpha_{T \cup S} \right] \\
&= \sum_{T \subseteq S_k, i \in T} \left[ \frac{\lambda_i}{\sum_{j \in T} \lambda_j} \sum_{S \subseteq \bigcup_{r=2}^{k-1} S_r} \beta_{T \cup S} \right] \\
&= \sum_{S \subseteq \bigcup_{r=2}^k S_r, i \in \bar{S}} \frac{\lambda_i}{\sum_{j \in \bar{S}} \lambda_j} \beta_S \\
&= \Phi_i^\omega(w) \\
&= \Phi_i^\omega(w^{L'}) = \mu_i^\omega(N \setminus S_1, w, L') = \mu_i^\omega(N \setminus S_1, z, L').
\end{aligned}$$

The third and the fifth equality use the fact that for every  $S$  with  $i \in \bar{S}$  there exists exactly one  $T \subseteq S_k$  and  $U \subseteq \bigcup_{r=1}^{k-1} S_r$  with  $S = T \cup U$ . Notice that it is possible for  $U$  to be empty and that it is possible that  $T$  is equal to  $\bar{S}$ . The fourth equality follows from lemma 3.5 and the seventh equality follows from lemma 3.3.  $\square$

Before we are able to characterize the Myerson values with weight systems we need to prove one more lemma. This lemma states that for every communication situation, where there are links only between players belonging to the lowest class, it holds that a solution concept satisfying component efficiency and class weighted fairness coincides with a Myerson value with weight system  $\omega$  for all players in this class.

**Lemma 3.7** *Let  $N \subseteq \Omega$ ,  $N$  finite, and  $(N, v, L) \in CS^N$  be a communication situation with  $L \subseteq \{\{i, j\} \mid i, j \in S_1\}$ . Let  $\gamma$  be a solution concept that satisfies CE and CWF with weights  $\lambda$ . Then it holds that  $\gamma(N, v, L) = \mu^\omega(N, v, L)$  with  $\omega = (\lambda, \Sigma)$ .*

**Proof:** According to lemma 3.1,  $\mu^\omega$  satisfies CE and CWF with weights  $\lambda$ . Now, suppose there exists a rule  $\gamma$  that satisfies CE and CWF with weights  $\lambda$  with  $\gamma \neq \mu^\omega$ . Let  $(N, v, L)$  be a communication situation with  $L \subseteq \{\{i, j\} \mid i, j \in S_1\}$  and a minimal number of links such that  $\gamma(N, v, L) \neq \mu^\omega(N, v, L)$ .

First let  $i \in N \setminus S_1$ . Since  $C_i(L) = \{i\}$  it follows from component efficiency that

$$\gamma_i(L) = v(\{i\}) = \mu_i^\omega(N, v, L).$$

Subsequently, let  $i \in S_1$  and  $\{i, j\} \in L$ . From CWF of  $\gamma$  we find:

$$\frac{1}{\lambda_i} (\gamma_i(L) - \gamma_i(L \setminus \{\{i, j\}\})) = \frac{1}{\lambda_j} (\gamma_j(L) - \gamma_j(L \setminus \{\{i, j\}\})).$$

Because  $\mu^\omega$  also satisfies CWF and using the minimality of  $L$  we derive

$$\begin{aligned}
\lambda_j \gamma_i(L) - \lambda_i \gamma_j(L) &= \lambda_j \gamma_i(L \setminus \{\{i, j\}\}) - \lambda_i \gamma_j(L \setminus \{\{i, j\}\}) \\
&= \lambda_j \mu_i^\omega(L \setminus \{\{i, j\}\}) - \lambda_i \mu_j^\omega(L \setminus \{\{i, j\}\}) \\
&= \lambda_j \mu_i^\omega(L) - \lambda_i \mu_j^\omega(L).
\end{aligned}$$

Thus,

$$\lambda_j(\gamma_i(L) - \mu_i^\omega(L)) = \lambda_i(\gamma_j(L) - \mu_j^\omega(L)).$$

This expression is valid for all directly connected pairs in  $S_1$ . Using transitivity, we find that the same expression holds for every pair  $\{s, t\}$  in a communication component. So, let  $C \in S_1/L$ ,  $i \in C$  and  $d(C) := \frac{1}{\lambda_i}(\gamma_i(L) - \mu_i^\omega(L))$ . For every  $j \in C$  we now have

$$\frac{1}{\lambda_j}(\gamma_j(L) - \mu_j^\omega(L)) = d(C).$$

CE of  $\gamma$  and  $\mu^\omega$  gives us

$$\sum_{j \in C} \gamma_j(L) = \sum_{j \in C} \mu_j^\omega(L) = v(C).$$

Thus,

$$0 = \sum_{j \in C} \{\gamma_j(L) - \mu_j^\omega(L)\} = \sum_{j \in C} \lambda_j d(C).$$

So,  $d(C) = 0$  for every communication component  $C \in S_1/L$ . This proves that  $\gamma_i(L) = \mu_i^\omega(L)$  for all  $i \in S_1$ . We conclude that  $\gamma(N, v, L) = \mu^\omega(N, v, L)$ .  $\square$

We can now characterize the Myerson value with weight systems.

**Theorem 3.1** Myerson values with weight systems satisfy component efficiency, class weighted fairness, higher class independency and class consistency. Furthermore, if a rule  $\gamma$  satisfies these four properties, then  $\gamma$  belongs to the class of Myerson values with weight systems.

**Proof:** Lemmas 3.1, 3.2 and 3.6 show that all Myerson values with weight systems satisfy CE, CWF, HCI and CC. Now, suppose  $\gamma$  is a rule that satisfies CE, CWF, HCI and CC. Let  $\lambda$  denote the weights for which  $\gamma$  satisfies CWF. Define  $\omega = (\lambda, \Sigma)$ . We will prove that  $\gamma = \mu^\omega$  by contradiction. Suppose  $\gamma \neq \mu^\omega$ . Let  $N \subseteq \Omega$  be finite and let  $(N, v, L) \in CS^N$  be a communication situation with a minimal number of links such that  $\gamma(N, v, L) \neq \mu^\omega(N, v, L)$ . If  $L = \emptyset$  then we find from CE of  $\gamma$  and  $\mu^\omega$  that  $\gamma(N, v, L) = \mu^\omega(N, v, L)$ . So,  $L \neq \emptyset$ . Let  $k$  denote the highest class with a player that forms a link, so

$$k := \max\{r \mid \exists s \in S_r, \exists t \in N : \{s, t\} \in L\}.$$

We will derive a contradiction by first showing for every player  $i$  in a lower hierarchical class than class  $k$  that  $\gamma_i(N, v, L) = \mu_i^\omega(N, v, L)$ . Subsequently we will show the same relation for every player  $i$  in a higher hierarchical class than class  $k$  and finally we will show for every player  $i$  in class  $k$  that  $\gamma_i(N, v, L) = \mu_i^\omega(N, v, L)$ .

Consider a class  $S_r$  with  $r < k$ . There exists at least one link in  $L$  with at least one of its end points in  $S_k$ . This means that this link is a member of  $L$  but not of  $L_{\leq r}$ , so  $|L_{\leq r}| < |L|$ . From HCI of  $\gamma$  and  $\mu^\omega$  and the minimality of  $L$  we conclude that for all  $i \in S_r$

$$\gamma_i(L) = \gamma_i(L_{\leq r}) = \mu_i^\omega(L_{\leq r}) = \mu_i^\omega(L).$$

Now consider a class  $S_r$ , with  $r > k$ , and a player  $i \in S_r$ . By definition of  $k$  player  $i$  is not connected to any other player. Hence, it follows from component efficiency that

$$\gamma_i(L) = v(\{i\}) = \mu_i^\omega(L).$$

Finally, we will consider class  $S_k$ . So, let player  $i \in S_k$ . Since  $\gamma$  and  $\mu^\omega$  satisfy class consistency

$$\gamma_i(N, v, L) = \gamma_i(N \setminus S_1, z, L') \text{ and } \mu_i^\omega(N, v, L) = \mu_i^\omega(N \setminus S_1, z, L').$$

The class-reduced communication situation has only players in classes  $S_2, \dots, S_m$ . We will denote  $(N \setminus S_1, z, L')$  by  $(N^2, v^2, L^2)$ . Since we have removed the players in the lowest class, this is a communication situation with  $m - 1$  classes. Notice that  $(N^2, L^2)$  has only links between players of the lowest  $k - 1$  classes.

We repeat this argument until player  $i$  is in the lowest class. This will be after  $k - 2$  more steps, and we will denote this communication situation by  $(N^k, v^k, L^k)$ . The communication graph  $(N^k, L^k)$  has only links between players in the lowest class. It now follows from lemma 3.7 that

$$\gamma_i(N^k, v^k, L^k) = \mu_i^\omega(N^k, v^k, L^k).$$

If we combine the results obtained so far we get

$$\begin{aligned} \gamma_i(N, v, L) &= \gamma_i(N^2, v^2, L^2) = \dots = \gamma_i(N^k, v^k, L^k) \\ &= \mu_i^\omega(N^k, v^k, L^k) = \dots = \mu_i^\omega(N^2, v^2, L^2) = \mu_i^\omega(N, v, L). \end{aligned}$$

We have now shown for all  $r \in \{1, \dots, m\}$  and all  $i \in S_r$  that  $\gamma_i(N, v, L) = \mu_i^\omega(N, v, L)$ . Thus  $\gamma(N, v, L) = \mu^\omega(N, v, L)$ .  $\square$

In the appendix it is shown that the four properties in the characterization above are logically independent.

## 4 An alternative axiomatic characterization

In this section we will provide an alternative axiomatic characterization of Myerson values with weight systems. In this characterization we use two properties that were used in the first characterization, component efficiency and higher class independency. Furthermore we introduce a third property, *class weighted balanced contributions*, a property inspired by balanced contributions (Myerson (1980)). Class weighted balanced contributions is stronger than class weighted fairness. We will show that if we replace class weighted fairness by class weighted balanced contributions in the characterization of the Myerson values with weight systems in section 3, we can drop the fourth axiom, class consistency.

In order to introduce the only new property in this section, class weighted balanced contributions, we introduce the following reduced set of links:

$$L_{-i} := \{\{j, k\} \in L \mid i \notin \{j, k\}\}.$$

This set of links is obtained from the original set of links by deleting the links in which player  $i$  is involved.

**Class weighted balanced contributions (CWBC) :** There exist weights  $\lambda = (\lambda_i)_{i \in \Omega}$  such that for all finite  $N \subseteq \Omega$ , all communication situations  $(N, v, L) \in CS^N$ , all  $k \in \{1, \dots, m\}$ , and all  $i, j \in S_k$  it holds that

$$\frac{\gamma_j(L) - \gamma_j(L_{-i})}{\lambda_j} = \frac{\gamma_i(L) - \gamma_i(L_{-j})}{\lambda_i}.$$

This property states that there exist weights such that in all communication situations two players can inflict the same weighted loss upon each other by breaking all links in which they are involved.

We will first show that a Myerson value with a weight system satisfies class weighted balanced contributions. After this we show that the class of Myerson values with weight systems can be characterized by the properties component efficiency, higher class independency and class weighted balanced contributions.

**Lemma 4.1** *The Myerson value with weight system  $\omega$  satisfies class weighted balanced contributions.*

**Proof:** Let  $N \subseteq \Omega$ ,  $N$  finite. Consider  $R_N$ , the set of all permutations of  $N$ . Furthermore, we consider the set of permutations  $R_\Sigma \subseteq R_N$  where for all  $k \in \{1, \dots, m-1\}$  it holds that the players of class  $S_k$  precede those of  $S_{k+1}$ , so  $R \in R_\Sigma$  can be identified with  $(R_1, \dots, R_m)$  where for all  $k \in \{1, \dots, m\}$  it holds that  $R_k$  is a permutation of  $S_k$ .



Furthermore we define the set of predecessors of a player according to permutation  $R$  by  $PR_i^R := \{j \in N : R^{-1}(j) \leq R^{-1}(i)\}$ .

Let  $(N, v, L) \in CS^N$ ,  $k \in \{1, \dots, m\}$ , and  $i, j \in S_k$ . According to *Kalai and Samet* (1988) there exists a probability measure  $P^\omega$  on  $R_\Sigma$  such that

$$\mu_i^\omega(N, v, L) = \sum_{S \subseteq N} \sum_{R \in R_\Sigma : PR_i^R = S} P^\omega(R) [v^L(S) - v^L(S \setminus \{i\})].$$

By using the notation  $R_\Sigma^i(S) := \{R \in R_\Sigma \mid PR_i^R = S\}$  and using the fact that  $\mu^\omega$  is component efficient we see that this expression is equal to

$$\sum_{S \subseteq N} \sum_{R \in R_\Sigma^i(S)} P^\omega(R) \left[ \sum_{h \in N} \mu_h^\omega(N, v, L(S)) - \sum_{h \in N} \mu_h^\omega(N, v, L(S)_{-i}) + v(\{i\}) \right].$$

Using this we find:

$$\begin{aligned} & \mu_i^\omega(N, v, L) - \mu_i^\omega(N, v, L_{-j}) \\ = & \sum_{S \subseteq N} \sum_{R \in R_\Sigma^i(S)} P^\omega(R) \left[ \sum_{h \in N} \mu_h^\omega(N, v, L(S)) - \sum_{h \in N} \mu_h^\omega(N, v, L(S)_{-i}) - \right. \\ & \left. \sum_{h \in N} \mu_h^\omega(N, v, L_{-j}(S)) + \sum_{h \in N} \mu_h^\omega(N, v, (L_{-j}(S))_{-i}) \right]. \end{aligned}$$

Since  $L(S)_{-i} = L_{-i}(S)$  and  $(L_{-j}(S))_{-i} = (L_{-i}(S))_{-j}$  we find that the expression between multiline brackets is symmetric in  $i$  and  $j$ . We denote this expression by  $K_{ij}(S)$ , so  $K_{ij}(S) = K_{ji}(S)$ . Note that  $K_{ij}(S) = 0$  for all  $S$  with  $\{i, j\} \not\subseteq S$ . In a similar way we find

$$\mu_j^\omega(N, v, L) - \mu_j^\omega(N, v, L_{-i}) = \sum_{S \subseteq N} \sum_{R \in R_\Sigma^j(S)} P^\omega(R) K_{ji}(S).$$

*Kalai and Samet* (1988) note that for all  $R \in R_\Sigma$  there exist probability measures  $P_1^\omega, \dots, P_m^\omega$  on  $R_{S_1}, \dots, R_{S_m}$  such that

$$P^\omega(R) = \prod_{r=1}^m P_r^\omega(R_r).$$

The claim in the appendix of *Dutta, Nouweland and Tijs* (1995) states that for all  $T \subseteq S_k$  with  $\{i, j\} \subseteq T$  it holds that

$$\sum_{R_k \in R_{S_k} : PR_i^{R_k} = T} P_k^\omega(R_k) = \frac{\lambda_i}{\lambda_j} \sum_{R_k \in R_{S_k} : PR_j^{R_k} = T} P_k^\omega(R_k).$$

Using this and multiplying with the sum over all possible permutations of all other classes of the probabilities of these permutations,

$$\sum_{R_{-k} \in R_{-S_k}} \prod_{r \neq k} P_r^\omega(R_r)$$

with  $R_{-k} = (R_1, \dots, R_{k-1}, R_{k+1}, \dots, R_m)$  and  $R_{-S_k} = \times_{r \neq k} R_{S_k}$  the cartesian product of the sets of all permutations of all classes except class  $k$ , gives us for all  $S \subseteq N$  with  $\{i, j\} \subseteq S$  that

$$\sum_{R \in R_\Sigma^i(S)} P^\omega(R) = \frac{\lambda_i}{\lambda_j} \sum_{R \in R_\Sigma^j(S)} P^\omega(R).$$

It follows that

$$\begin{aligned} \mu_i^\omega(N, v, L) - \mu_i^\omega(N, v, L_{-j}) &= \sum_{S \subseteq N} \sum_{R \in R_\Sigma^i(S)} P^\omega(R) K_{ij}(S) \\ &= \sum_{S \subseteq N} \frac{\lambda_i}{\lambda_j} \sum_{R \in R_\Sigma^j(S)} P^\omega(R) K_{ij}(S) \\ &= \frac{\lambda_i}{\lambda_j} \sum_{S \subseteq N} \sum_{R \in R_\Sigma^j(S)} P^\omega(R) K_{ji}(S) \\ &= \frac{\lambda_i}{\lambda_j} [\mu_j^\omega(N, v, L) - \mu_j^\omega(N, v, L_{-i})]. \end{aligned}$$

We conclude that  $\mu^\omega$  satisfies class weighted balanced contributions.  $\square$

Note that class weighted balanced contributions implies class weighted fairness. This follows easily by noting that for an allocation  $\gamma$  that satisfies class weighted balanced contributions it holds for all  $i, j \in N$  that

$$\begin{aligned} \frac{\gamma_i(L) - \gamma_i(L \setminus \{\{i, j\}\})}{\lambda_i} &= \frac{\gamma_i(L) - \gamma_i(L_{-j}) - [\gamma_i(L \setminus \{\{i, j\}\}) - \gamma_i(L_{-j})]}{\lambda_i} \\ &= \frac{\gamma_j(L) - \gamma_j(L_{-i}) - [\gamma_j(L \setminus \{\{i, j\}\}) - \gamma_j(L_{-i})]}{\lambda_j} = \frac{\gamma_j(L) - \gamma_j(L \setminus \{\{i, j\}\})}{\lambda_j}, \end{aligned}$$

where the second equality follows from class weighted balanced contributions of  $\gamma$  and the notion that  $L_{-i} = (L \setminus \{\{i, j\}\})_{-i}$  and  $L_{-j} = (L \setminus \{\{i, j\}\})_{-j}$ .

The next theorem shows that if we replace class weighted fairness by class weighted balanced contributions in the characterization of Myerson values with weight systems in section 3, we do not need class consistency to characterize the Myerson values with weight systems.

**Theorem 4.1** Myerson values with weight systems satisfy component efficiency, higher class independency and class weighted balanced contributions. Furthermore, if a rule  $\gamma$  satisfies these three properties, then  $\gamma$  belongs to the class of Myerson values with weight systems.

**Proof :** We have already shown that all Myerson values with weight systems satisfy CE, HCI and CWBC. Now suppose  $\gamma$  is a rule that also satisfies these three properties. Let  $\lambda$  denote the weights for which  $\gamma$  satisfies CWBC. Define  $\omega = (\lambda, \Sigma)$ .

Let  $N \subseteq \Omega$ ,  $N$  finite,  $(N, v)$  a cooperative game,  $k \in \{1, \dots, m\}$ , and  $i \in S_k$ . Denote with  $C_i(L_{\leq k})$  the component in the graph  $(N, L_{\leq k})$  containing  $i$ . We will prove that  $\gamma_i = \mu_i^\omega$  by induction to the number of players in  $|C_i(L_{\leq k}) \cap S_k|$ .

If  $C_i(L_{\leq k}) \cap S_k = \{i\}$ , it follows from HCI and CE that

$$\sum_{j \in C_i(L_{\leq k}) \setminus S_k} \mu_j^\omega(N, v, L) = v^L(C_i(L_{\leq k}) \setminus S_k) = \sum_{j \in C_i(L_{\leq k}) \setminus S_k} \gamma_j(N, v, L),$$

since  $C_i(L_{\leq k}) \setminus \{i\}$  is the union of a number of components in the graph  $(N, L_{\leq k-1})$ . Furthermore, from CE and HCI it follows that

$$\sum_{j \in C_i(L_{\leq k})} \mu_j^\omega(N, v, L) = v(C_i(L_{\leq k})) = \sum_{j \in C_i(L_{\leq k})} \gamma_j(N, v, L).$$

It now follows directly that  $\mu_i^\omega(N, v, L) = \gamma_i(N, v, L)$ .

Suppose we have already proved that  $\mu_i^\omega(N, v, L) = \gamma_i(N, v, L)$  for all  $(N, v, L)$  with  $|C_i(L_{\leq k}) \cap S_k| \leq l$ . Let  $(N, v, L) \in CS^N$  be such that  $|C_i(L_{\leq k}) \cap S_k| = l + 1$ . For notational convenience we denote  $C_i(L_{\leq k}) \cap S_k = \{1, \dots, l + 1\}$ . In the following system of linear equations the first equation follows from CE and HCI. The second set of equations follows from CWBC.

$$\sum_{j \in C_i(L_{\leq k}) \cap S_k} \gamma_j(L) = v(C_i(L_{\leq k})) - v^L(C_i(L_{\leq k}) \setminus S_k),$$

and for each  $j \in (C_i(L_{\leq k}) \setminus \{1\}) \cap S_k$ :

$$\frac{\gamma_j(L) - \gamma_j(L_{-1})}{\lambda_j} = \frac{\gamma_1(L) - \gamma_1(L_{-j})}{\lambda_1}.$$

This system can be rewritten using the induction hypothesis as:

$$\sum_{j \in C_i(L_{\leq k}) \cap S_k} \gamma_j(N, v, L) = v(C_i(L_{\leq k})) - v^L(C_i(L_{\leq k}) \setminus S_k),$$

and for each  $j \in (C_i(L_{\leq k}) \setminus \{1\}) \cap S_k$  :

$$\lambda_1 \gamma_j(N, v, L) - \lambda_j \gamma_1(N, v, L) = \lambda_1 \mu_j^\omega(N, v, L_{-1}) - \lambda_j \mu_1^\omega(N, v, L_{-j}).$$

Since the system is determined by a square non-singular matrix we have that this system has a unique solution. Since  $\mu^\omega$  satisfies CE, HCI and CWBC it is a solution of this system of equations. Consequently,  $\gamma = \mu^\omega$ .  $\square$

In the appendix it is shown that the three properties in the characterization above are logically independent.

## Remark

*Borm, Owen and Tijs* (1992) introduced the *position value* for communication situations without a hierarchical player partition. First they introduced the *link game*  $(L, v^N)$  which assigns to every  $A \subseteq L$  the corresponding value of the grand coalition,  $v^N(A) = \sum_{C \in N/A} v(C)$ . The *link value* assigns to every communication link the Shapley value of the link game. The position value then equally splits the value of a link between the two players who form the link.

It is possible to extend the position value to an allocation rule for communication situations with a hierarchical player partition. The *extended position value* divides the value of a link equally between the two players who form the link if they are in the same class. Otherwise the value is attributed to the player who is in the higher class.

The extended position value can be characterized in a similar way to *Borm et al.* (1992). The properties *component efficiency*, *additivity*, and *superfluous link property* used by *Borm et al.* (1992) in the characterization of the position value are used unchanged in the characterization of the extended position value. Their fourth property, *link anonymity*, needs to be generalized in a straightforward way. This generalized property could be called *class link anonymity*.

## Appendix

In this appendix we will prove the irredundancy of the properties used in the two characterizations in this paper.

First we will show that the four properties that we used to characterize the Myerson values with weight systems in theorem 3.1 are logically independent. We do this by

providing for each of the four properties an example of an allocation rule that does not satisfy this specific property, but that does satisfy the three remaining properties.

The irredundancy of component efficiency follows immediately if we consider the allocation rule that attributes zero to every player in every communication situation.

To prove the irredundancy of class weighted fairness we introduce a new allocation rule  $\sigma$ . For all components in the reduced graph  $(S_1, L_{\leq 1})$  this rule divides equally the gain of a component among the players of this component. When the lowest  $k - 1$  classes are handled we determine for every player  $i$  in  $S_k$  the component of this player in the graph  $(N, L_{\leq k})$ . We will denote this component by  $C_i(L_{\leq k})$ . The players in the intersection of this component with class  $k$  divide equally the amount that results when the payoff attributed to the players in  $C_i(L_{\leq k}) \setminus S_k$  is subtracted from the value of the component  $C_i(L_{\leq k})$ . In formula, for all finite  $N \subseteq \Omega$ , all  $(N, v, L) \in CS^N$ , all  $k \in \{1, \dots, m\}$ , and all  $i \in S_k$

$$\sigma_i(N, v, L) := \frac{1}{|C_i(L_{\leq k}) \cap S_k|} \left( v^L(C_i(L_{\leq k})) - v^L(C_i(L_{\leq k}) \setminus S_k) \right).$$

**Theorem A.1** The allocation rule  $\sigma$  satisfies component efficiency, higher class independency and class consistency.

**Proof:** Let  $N \subseteq \Omega$  be finite and  $(N, v)$  a cooperative game. Higher class independency of  $\sigma$  follows by noting that  $v(C_i(L_{\leq k})) = v^{L_{\leq k}}(C_i(L_{\leq k}))$  and  $v(C_i(L_{\leq k}) \setminus S_k) = v^{L_{\leq k}}(C_i(L_{\leq k}) \setminus S_k)$  for all  $(N, L)$ , all  $k \in \{1, \dots, m\}$ , and all  $i \in S_k$ .

We will prove that  $\sigma$  satisfies component efficiency by induction. Let  $C$  be a component of  $(N, L)$ . If  $\max\{k \mid C \cap S_k \neq \emptyset\} = 1$ , then

$$\sum_{i \in C} \sigma_i(L) = \sum_{i \in C} \frac{1}{|C|} (v(C) - v(\emptyset)) = v(C).$$

Suppose we already proved for all components  $C$  with  $\max\{k \mid C \cap S_k \neq \emptyset\} \leq l$  that  $\sum_{i \in C} \sigma_i(L) = v(C)$ . Let  $(N, v, L) \in CS^N$  and  $C \in N/L$  be such that  $\max\{k \mid C \cap S_k \neq \emptyset\} = l + 1$ . Then,

$$\sum_{i \in C \cap S_{l+1}} \sigma_i(L) = v(C) - v^L(C \setminus S_{l+1}).$$

By the induction hypothesis and the higher class independency of  $\sigma$  it follows that

$$\begin{aligned} \sum_{i \in C} \sigma_i(L) &= v(C) - v^L(C \setminus S_{l+1}) + \sum_{i \in C \setminus S_{l+1}} \sigma_i(L_{\leq l}) \\ &= v(C) - v^L(C \setminus S_{l+1}) + v^L(C \setminus S_{l+1}) \\ &= v(C). \end{aligned}$$

We conclude that  $\sigma$  satisfies component efficiency.

To prove class consistency let  $i \in S_k$  with  $k > 1$ . Notice that  $v^L(C_i(L_{\leq k})) = v^L(C_i(L_{\leq k}) \cup S_1) - v^L(S_1 \setminus C_i(L_{\leq k}))$ , since the partition of  $C_i(L_{\leq k}) \cup S_1$  in  $C_i(L_{\leq k})$  and  $S_1 \setminus C_i(L_{\leq k})$  does not split up any component of  $(C_i(L_{\leq k}) \cup S_1, L)$ . Using this we find

$$\begin{aligned}
\sigma_i(N, v, L) &= \frac{1}{|C_i(L_{\leq k}) \cap S_k|} \left\{ v^L(C_i(L_{\leq k})) - v^L(C_i(L_{\leq k}) \setminus S_k) \right\} \\
&= \frac{1}{|C_i(L_{\leq k}) \cap S_k|} \left\{ v^L(C_i(L_{\leq k})) - v^L(C_i(L_{\leq k}) \setminus S_k) \right. \\
&\quad \left. + v^L(S_1 \setminus C_i(L_{\leq k})) - v^L(S_1 \setminus C_i(L_{\leq k})) \right\} \\
&= \frac{1}{|C_i(L_{\leq k}) \cap S_k|} \left\{ v^L(C_i(L_{\leq k}) \cup S_1) - v^L((C_i(L_{\leq k}) \setminus S_k) \cup S_1) \right\} \\
&= \frac{1}{|C_i(L_{\leq k}) \cap S_k|} \left\{ v^L(C_i(L_{\leq k}) \cup S_1) - v^L(S_1) \right. \\
&\quad \left. - v^L((C_i(L_{\leq k}) \setminus S_k) \cup S_1) + v^L(S_1) \right\} \\
&= \frac{1}{|C_i(L_{\leq k}) \cap S_k|} \left\{ z(C_i(L_{\leq k}) \setminus S_1) - z((C_i(L_{\leq k}) \setminus S_1) \cup S_k) \right\} \\
&= \frac{1}{|C_i((L')_{\leq k}) \cap S_k|} \left\{ z^{L'}(C_i((L')_{\leq k})) - z^{L'}((C_i((L')_{\leq k}) \setminus S_k) \right\} \\
&= \sigma_i(N \setminus S_1, z, L').
\end{aligned}$$

In the fifth equality we use  $v^L(S_1) = \sum_{i \in S_1} \sigma_i(S \cup S_1, v, L)$  for all  $S \subseteq N \setminus S_1$ . The sixth equality follows from lemma 3.3, the notion that  $(N \setminus S_1, w)$  and  $(N \setminus S_1, z)$  coincide, and the fact that  $C_i(L_{\leq k}) \setminus S_1 = C_i((L')_{\leq k})$ .  $\square$

The following example shows that the allocation rule  $\sigma$  does not coincide with any Myerson value with a weight system and hence,  $\sigma$  does not satisfy class weighted fairness.

**Example A.1** Consider the communication situation  $(N, v, L) \in CS^N$  with  $N = S_1 = \{1, 2\}$ ,  $L = \{\{1, 2\}\}$  and  $v = u_{\{1\}}$ . We see from this that  $\sigma_1(N, v, L) = \sigma_2(N, v, L) = \frac{1}{2}$  while for all  $\omega = (\lambda, (S_1))$  with  $\lambda \in \mathbb{R}_{++}^2$ :  $\mu_1^\omega(N, v, L) = 1$  and  $\mu_2^\omega(N, v, L) = 0$ .  $\square$

We proceed by showing the irredundancy of class consistency. We introduce a class of allocation rules that satisfy component efficiency, class weighted fairness and higher class independency, but not class consistency. For this, we determine for every player in the highest class the component of this player when links between players in the highest class are not considered. Further, we determine the difference between the value of this component in the situation using only links that are not within the highest class and the value of the same component except the players in the highest class. This difference is then divided equally between the players of the intersection of this component with the

highest class. The non-attributed values are then divided in the same way as a Myerson value with a weight system.

To describe this value formally we need some more notation. Let

$$L_{<m} := \{\{i, j\} \mid \{i, j\} \in L, \{i, j\} \not\subseteq S_m\}$$

and for all  $i \in S_m$

$$\alpha_i := \frac{v(C_i(L_{<m})) - v(C_i(L_{<m}) \setminus S_m)}{|C_i(L_{<m}) \cap S_m|}.$$

Now define for all  $\omega = (\lambda, \Sigma)$  with  $\lambda \in \mathbf{R}_{++}^\Omega$  the following allocation rule. For all finite  $N \subseteq \Omega$  and all  $(N, v, L) \in CS^N$

$$\rho_i^\omega(N, v, L) := \begin{cases} \mu_i^\omega(N, v, L) & , i \notin S_m \\ \alpha_i + \mu_i^\omega(N, v^L - v^{L_{<m}}, L) & , i \in S_m \end{cases}.$$

The following theorem shows that  $\rho^\omega$  satisfies component efficiency, class weighted fairness and higher class independency.

**Theorem A.2** The allocation rule  $\rho^\omega$  satisfies component efficiency, class weighted fairness and higher class independency.

**Proof:** Let  $N \subseteq \Omega$ ,  $N$  finite, and  $(N, v, L) \in CS^N$ . Higher class independency of  $\rho^\omega$  follows immediately from the higher class independency of the Myerson value with weight system  $\omega$ .

To prove class weighted fairness notice that  $L_{<m} = (L \setminus \{\{i, j\}\})_{<m}$  for all  $\{i, j\} \subseteq S_m$ , so it holds for all  $r \in S_m$  that neither  $\alpha_r$  nor  $v^L - v^{L_{<m}}$  change as a result of the deletion of link  $\{i, j\}$ . Hence class weighted fairness of  $\rho^\omega$  follows from class weighted fairness of the Myerson value with weight system  $\omega$ .

We still have to show that  $\rho^\omega$  satisfies component efficiency. Let  $C$  be a component of the graph  $(N, L)$ , so  $C \in N/L$ , then since

$$\sum_{i \in C \cap S_m} \alpha_i = v^{L_{<m}}(C) - v^{L_{<m}}(C \setminus S_m) = \sum_{i \in C \cap S_m} \mu_i^\omega(N, v^{L_{<m}}, L),$$

it holds that

$$\begin{aligned} \sum_{i \in C \cap S_m} \rho_i^\omega(N, v, L) &= \sum_{i \in C \cap S_m} \alpha_i + \sum_{i \in C \cap S_m} \mu_i^\omega(N, v^L - v^{L_{<m}}, L) \\ &= \sum_{i \in C \cap S_m} \mu_i^\omega(N, v^{L_{<m}}, L) + \sum_{i \in C \cap S_m} \mu_i^\omega(N, v^L - v^{L_{<m}}, L) \\ &= \sum_{i \in C \cap S_m} \mu_i^\omega(N, v^L, L), \end{aligned}$$

where the third equality follows from the additivity of  $\mu^\omega$  which follows from the additivity of  $\Phi_\omega$ .

Using this and the fact that

$$\sum_{i \in C \cap S_m} \mu_i^\omega(N, v^L, L) = \sum_{i \in C \cap S_m} \mu_i^\omega(N, v, L),$$

since  $(v^L)^L = v^L$ , we obtain

$$\begin{aligned} \sum_{i \in C} \rho_i^\omega(N, v, L) &= \sum_{i \in C \setminus S_m} \mu_i^\omega(N, v, L) + \sum_{i \in C \cap S_m} \mu_i^\omega(N, v, L) \\ &= \sum_{i \in C} \mu_i^\omega(N, v, L) \\ &= v(C), \end{aligned}$$

so  $\rho^\omega$  satisfies component efficiency.  $\square$

To prove that  $\rho^\omega$  does not satisfy class consistency consider the following example.

**Example A.2** Consider the communication situation  $(N, v, L) \in CS^N$  with  $N = \{1, 2, 3, 4\}$ ,  $L = \{\{1, 2\}, \{1, 3\}, \{2, 4\}\}$  and  $v = 2u_{\{2,3\}}$ . Further the partition in classes is as follows:  $S_1 = \{1, 2\}$  and  $S_2 = \{3, 4\}$ . Since  $v^L = v^{L_{<m}}$  it follows that  $\alpha_3 = \alpha_4 = 1$  and for all  $\omega = (\lambda, (S_1, S_2))$  with  $\lambda \in \mathbf{R}_{++}^N$

$$\rho_3^\omega(N, v, L) = \rho_4^\omega(N, v, L) = 1 + 0 = 1.$$

Since

$$\mu_3^\omega(N, v, L) = 2 \text{ and } \mu_4^\omega(N, v, L) = 0,$$

we conclude that  $\rho^\omega$  does not coincide with any Myerson value with weight system. Hence,  $\rho^\omega$  does not satisfy class consistency.  $\square$

Finally we will introduce a class of solution concepts that satisfy component efficiency, class weighted fairness and class consistency. Further we will show that these rules do not satisfy higher class independency. Every rule in this class is based on a Myerson value with a weight system. However, the values attributed by the Myerson value with a weight system to the players in the same class and the same component according to a certain restricted graph associated with that class is now divided equally between these players.

To describe this value formally define  $L^0 := L$  and  $L^1, \dots, L^m$  are the sets of links as defined on page 8. Further, we define

$$L'^k := L^{k-1} \setminus (L^{k-1})_{\leq k} \text{ for all } k \in \{1, \dots, m\}.$$



Since the lowest class with a player that forms a link in  $L^{k-1}$  is class  $k$ , the set  $L^k$  contains the links in the set  $L^{k-1}$  except the links between two players in  $S_k$ .

Let  $\omega = (\lambda, \Sigma)$  with  $\lambda \in \mathbb{R}_{++}^\Omega$ . Now define the allocation rule  $\delta^\omega$ . For all finite  $N \subseteq \Omega$ , all  $(N, v, L) \in CS^N$ , all  $k \in \{1, \dots, m\}$ , and all  $i \in S_k$

$$\delta_i^\omega(N, v, L) := \frac{\sum_{j \in C_i(L^k) \cap S_k} \mu_j^\omega(N, v, L^k)}{|C_i(L^k) \cap S_k|} + \mu_i^\omega(N, v, L) - \mu_i^\omega(N, v, L^k).$$

Theorem A.3 shows that this rule satisfies component efficiency, class weighted fairness and class consistency

**Theorem A.3** The allocation rule  $\delta^\omega$  satisfies component efficiency, class weighted fairness and class consistency.

**Proof:** Let  $N \subseteq \Omega$  be finite, let  $(N, v, L) \in CS^N$  be a communication situation, and let  $\omega = (\lambda, \Sigma)$  with  $\lambda \in \mathbb{R}_{++}^\Omega$ . First we will show that  $\delta^\omega$  satisfies component efficiency. Let  $C \in N/L$  be a communication component. Denote for  $k \in \{1, \dots, m\}$  the intersection of  $C$  with class  $k$  by  $C_k := C \cap S_k$ . Further, we determine for all  $k \in \{1, \dots, m\}$  the partition of  $C_k$  in  $\{C_k^1, \dots, C_k^{m_k}\}$  such that if  $i \in C_k^l$  then  $C_i(L^k) \cap S_k = C_k^l$ .

With these definitions it holds for all  $k \in \{1, \dots, m\}$  and all  $l \in \{1, \dots, m_k\}$  that

$$\begin{aligned} \sum_{i \in C_k^l} \frac{\sum_{j \in C_i(L^k) \cap S_k} \mu_j^\omega(N, v, L^k)}{|C_i(L^k) \cap S_k|} &= \sum_{i \in C_k^l} \frac{\sum_{j \in C_k^l} \mu_j^\omega(N, v, L^k)}{|C_k^l|} \\ &= \sum_{j \in C_k^l} \mu_j^\omega(N, v, L^k). \end{aligned}$$

From this and the component efficiency of the Myerson value with weight system  $\omega$  it follows that

$$\begin{aligned} \sum_{i \in C} \delta_i^\omega(N, v, L) &= \sum_{k \in \{1, \dots, m\}} \sum_{l=1}^{m_k} \sum_{i \in C_k^l} \delta_i^\omega(N, v, L) \\ &= \sum_{k \in \{1, \dots, m\}} \sum_{l=1}^{m_k} \sum_{i \in C_k^l} \left( \mu_i^\omega(N, v, L^k) + \mu_i^\omega(N, v, L) - \mu_i^\omega(N, v, L^k) \right) \\ &= \sum_{i \in C} \mu_i^\omega(N, v, L) \\ &= v(C). \end{aligned}$$

This proves that  $\delta^\omega$  satisfies component efficiency.

To prove class weighted fairness note that for all  $k \in \{1, \dots, m\}$  and all  $i, j \in S_k$

$$\begin{aligned} (L \cup \{i, j\})^{k-1} &= (L \cup \{i, j\})^{>k-1} \cup (L \cup \{i, j\})^{*k-1} \\ &= (L^{>k-1} \cup \{i, j\}) \cup L^{*k-1} \\ &= L^{k-1} \cup \{i, j\}. \end{aligned}$$

Since  $\{i, j\} \in ((L \cup \{i, j\})^{k-1})_{\leq k}$  it holds that

$$\begin{aligned} (L \cup \{i, j\})'^k &= (L \cup \{i, j\})^{k-1} \setminus ((L \cup \{i, j\})^{k-1})_{\leq k} \\ &= L^{k-1} \setminus (L^{k-1})_{\leq k} \\ &= L'^k. \end{aligned}$$

This implies that  $C_i(L'^k) = C_i((L \cup \{i, j\})'^k)$ . Now, class weighted fairness of  $\delta^\omega$  follows from class weighted fairness of the Myerson value with weight system  $\omega$

Finally we need to show that  $\delta^\omega$  satisfies class consistency. Let  $k \in \{2, \dots, m\}$ , then it follows that

$$\begin{aligned} (L'^k)' &= (L^{k-1} \setminus (L^{k-1})_{\leq k})' = (L^{k-1} \setminus (L^{k-1})_{\leq k})^{>1} \cup (L^{k-1} \setminus (L^{k-1})_{\leq k})^{*1} \\ &= (L^{k-1} \setminus (L^{k-1})_{\leq k}) \cup \emptyset = L'^k, \end{aligned}$$

since  $k > 1$ . Further it follows that

$$L'^k = L^{k-1} \setminus (L^{k-1})_{\leq k} = (L')^{k-1} \setminus ((L')^{k-1})_{\leq k} = (L')'^k.$$

We conclude that  $(L'^k)' = (L')'^k$ . Since the Myerson value with weight systems satisfies class consistency we find for all  $i \in S_k$  that

$$\begin{aligned} \delta_i^\omega(N, v, L) &= \frac{\sum_{j \in C_i(L'^k) \cap S_k} \mu_j^\omega(N, v, L'^k)}{|C_i(L'^k) \cap S_k|} + \mu_i^\omega(N, v, L) - \mu_i^\omega(N, v, L'^k) \\ &= \frac{\sum_{j \in C_i(L'^k) \cap S_k} \mu_j^\omega(N \setminus S_1, z, (L'^k)')}{|C_i(L'^k) \cap S_k|} \\ &\quad + \mu_i^\omega(N \setminus S_1, z, L') - \mu_i^\omega(N \setminus S_1, z, (L'^k)') \\ &= \frac{\sum_{j \in C_i((L')'^k) \cap S_k} \mu_j^\omega(N \setminus S_1, z, (L')'^k)}{|C_i((L')'^k) \cap S_k|} \\ &\quad + \mu_i^\omega(N \setminus S_1, z, L') - \mu_i^\omega(N \setminus S_1, z, (L')'^k) \\ &= \delta_i^\omega(N \setminus S_1, z, L'). \end{aligned}$$

We conclude that  $\delta^\omega$  satisfies class consistency.  $\square$

The following example shows that the allocation rule  $\delta^\omega$  is not equal to any Myerson value with a weight system. This implies that  $\delta^\omega$  does not satisfy higher class independency.

**Example A.3** Let  $(N, v, L) \in CS^N$  with player set  $N = \{1, 2, 3\}$ , characteristic function  $v = u_{\{1\}}$  and the set of links  $L = \{\{1, 3\}, \{2, 3\}\}$ . Further let  $\omega = (\lambda, (\{1, 2\}, \{3\}))$  with  $\lambda \in \mathbb{R}_{++}^N$ . Since  $v^L = v$  it follows immediately that  $\mu_1^\omega(N, v, L) = 1$ ,  $\mu_2^\omega(N, v, L) = 0$  and  $\mu_3^\omega(N, v, L) = 0$ .

Since  $L^1 = L$  it follows that  $C_1(L^1) \cap S_1 = C_2(L^1) \cap S_1 = \{1, 2\}$  which implies that  $\delta_1^\omega(N, v, L) = \frac{1}{2} + 1 - 1 = \frac{1}{2}$ ,  $\delta_2^\omega(N, v, L) = \frac{1}{2} + 0 - 0 = \frac{1}{2}$  and  $\delta_3^\omega(N, v, L) = 0 + 0 - 0 = 0$ . We conclude that there does not exist any Myerson value with weight system that is equal to  $\delta^\omega$ .  $\square$

We conclude that the four properties used in Theorem 3.1 are irredundant.

To conclude this appendix, we will show that the three properties used in Theorem 4.1 are logically independent. The irredundancy of component efficiency follows immediately if we consider the allocation rule that attributes zero to every player in every communication situation. Since class weighted balanced contributions implies class weighted fairness, the irredundancy of class weighted balanced contributions follows if we consider the allocation rule  $\sigma$  that we introduced to show the irredundancy of class weighted fairness in the first axiomatic characterization.

Now, we introduce an allocation rule that satisfies component efficiency and class weighted balanced contributions, but that does not satisfy higher class independency. Before we can introduce this rule we need some more notations. We define a weak order  $\Sigma^{-1}$  such that for all  $i, j \in \Omega$ ,  $(i, j) \in \Sigma^{-1}$  if and only if  $(j, i) \in \Sigma$ . We will call  $\Sigma^{-1}$  the reverse hierarchical order. Given a weight system  $\omega = (\lambda, \Sigma)$  we introduce a weight system  $\bar{\omega} := (\lambda, \Sigma^{-1})$ .

Now we define for all  $\omega = (\lambda, \Sigma)$  with  $\lambda \in \mathbb{R}_{++}^\Omega$  an allocation rule  $\xi^\omega$  as follows. For all finite  $N \subseteq \Omega$  and all  $(N, v, L) \in CS^N$

$$\xi^\omega(N, v, L) = \frac{1}{2} \left( \mu^\omega(N, v, L) + \mu^{\bar{\omega}}(N, v, L) \right).$$

From component efficiency and class weighted balanced contributions of the Myerson values with weight systems it follows that  $\xi^\omega$  satisfies component efficiency and class weighted balanced contributions.

The following example shows that  $\xi^\omega$  does not belong to the class of Myerson values with weight systems.

**Example A.4** Consider the communication situation  $(N, v, L) \in CS^N$  with  $N = \{1, 2\}$ ,  $L = \{\{1, 2\}\}$  and  $v = u_{\{1, 2\}}$ . Further the partition in classes is as follows:  $S_1 = \{1\}$  and  $S_2 = \{2\}$ . Since  $v^L = v$  it follows immediately for all  $\omega = (\lambda, (S_1, S_2))$  with  $\lambda \in \mathbb{R}_{++}^2$  that

$$\mu_1^\omega(N, v, L) = 0 \text{ and } \mu_2^\omega(N, v, L) = 1$$

Furthermore it holds that

$$\mu_1^{\bar{\omega}}(N, v, L) = 1 \text{ and } \mu_2^{\bar{\omega}}(N, v, L) = 0,$$

so we find

$$\xi_1^\omega(N, v, L) = \xi_2^\omega(N, v, L) = \frac{1}{2}.$$

We conclude that  $\xi^\omega$  does not coincide with any Myerson value with weight system, so  $\xi^\omega$  does not satisfy higher class independency.  $\square$

We conclude that the three properties used in Theorem 4.1 are logically independent.

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